RESONANT GROWTH OF PERTURBATIONS IN A SUPERSONIC JET

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The dynamics of the growth of perturbations downstream in a supersonic jet has not been extensively studied even though a cursory acquaintance with the literature shows interesting and diverse phenomena which can be explained with the help of the slightly nonlinear theory of hydrodynamic stability.

It is known that in a jet there is a wide spectrum of instabilities which interact with one another. For example, two-dimensional spatial instabilities can occur near the origin of the jet, where the geometry of the flow is one-dimensional. Perturbations for which the existence of a radius of curvature of the flow is essential (cylindrical waves) show up in the amplitude–frequency spectrum of fluctuations. Examples are axisymmetric (azimuthal number $n = 0$) and azimuthal waves taking the form of simple ($n = \pm 1$), double ($n = \pm 2$), and higher-order helixes. Normally helixes of the lowest mode occur and in the early stages of the evolution the left ($n = +1$) and right ($n = -1$) helixes are synchronized in amplitude and phase, forming beats.

The flow in a supersonic jet is such that perturbations grow over a background of small-scale turbulence and undoubtedly at a certain stage large-scale oscillations may be associated with this background. There always exist perturbations with negative group velocity and hence they complete a feedback loop, inducing oscillations near the origin of the jet. Recent data on longitudinally spiralling flow in the initial regions of jets suggests that the latter should be taken as hypothetical triggering objects until proof is obtained that they do not participate in the dynamics of travelling waves.

Jet-column mode oscillations are undoubtedly the most important. They carry energy and propagate over large longitudinal distances. Hot-wire anemometer data on the evolution of waves are associated with these perturbations. Spectrogram analysis [1] shows that the amplitude–frequency spectrum of fluctuations in the initial region of the jet shows two peaks with frequencies corresponding to a relatively narrow range of Struckel numbers and the ratio of the frequencies at these peaks is approximately 1 to 2. Fluctuations at the lower frequency dominate near the nozzle, the intensities corresponding to the two peaks become equal at a certain distance from the nozzle, and at further distances the higher-frequency oscillations begin to dominate. Further downstream strongly nonlinear processes begin and the spectrum broadens and the two peaks join. This type of perturbation growth is observed over a wide range of Mach numbers (from 1.4 to 3). The nature of the perturbation corresponding to each peak remains an open question, as an unambiguous answer to this question has not yet been obtained.

In the present paper we consider the region corresponding to the start of nonlinear effects where oscillations in the form of two separate peaks evolve. Among the known mechanisms of multiple-frequency wave interactions the simplest is the model of a three-wave resonant system interacting nonlinearly to lowest order with respect to the initial linear processes. The study of interactions of this type was begun in [2] and has been extended and applied successfully to transition stages in subsonic boundary layers [3, 4] and Poiseuille flow [5]. Therefore, it is logical to consider the possible conditions under which resonant mechanisms can exist in the evolution of wave processes in jets.

The basic idea of the slightly nonlinear approach in stability problems is the solution of simple equations describing the behavior of the wave amplitudes in space and time. Then the behavior of quantities undefined in linear problems can be determined qualitatively and corrections in the evolution of the perturbations can be calculated. In the case of jet-column modes the resonant triplet should consist of an axisymmetric mode (the analog of a plane wave or higher harmonics) and a pair of azimuthal oscillations (right and left helixes

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of the same order in n at a lower frequency). This pair is the analog of skew waves (subharmonics) in boundary layers.

We note that the specific feature of free jet flow is the presence of an inflection point in the velocity profile. Because many of the relations important in semi-infinite viscous flow do not hold (in particular, the relations between the viscous and nonlinear terms), many of the restrictions on the maximum amplitudes that can be considered in the slightly nonlinear theory are no longer true (in [1] the mass velocity distribution shows that the fluctuations accurately obey the linear theory even when their relative amplitude exceeds 10%, which is several times larger than the linear limit in free shear flow at subsonic velocities). Hence the conditions under which nonlinear processes can compete with linear processes are not completely known and the problem of perturbations in jets is convenient in studying this question.

1. Basic Equations. We consider the wave dynamics in a compressible, non-heat-conducting gas flowing from a circular nozzle of radius $\bar{r}_0$ with exit velocity $\bar{W}_0$ (characteristic parameters). Neglecting expansion of the jet cross section, the velocity field can be written in the form $u_i = |\varepsilon u', \varepsilon v', W + \varepsilon w'|$ (u', v', w' are the transverse, azimuthal, and longitudinal components of the fluctuation field in the radial r, azimuthal φ, and longitudinal z directions, and W is the longitudinal component of the average velocity).

In the initial region of the jet the dimensionless W profile in the mixing layer δ is approximately $W(r) = \exp(-0.693714\eta^2)$ [$\eta = 2(r - r_i)/\delta$ is the self-similar variable transverse coordinate r with the start of the mixing layer $r_i = 1 - \delta/2$)]. The factor $z(\delta)$ is given by the Abramovich formula [6] $z = \delta/b(M)$ with $b = 0.2281$, $M = 1.5$ (M is the Mach number at the nozzle).

The linearized Euler's equations for the perturbations are

$$\begin{gathered}
\varepsilon(u_t' + Wu_z' + p_r'/\rho_0) + \varepsilon^2 d_1 = 0, \quad \varepsilon(v_t' + Wv_z' + p_\varphi'/\rho_0 r) + \varepsilon^2 d_2 = 0, \\
\varepsilon(w_t' + Ww_z' + W_r u' + p_z'/\rho_0) + \varepsilon^2 d_3 = 0, \\
\varepsilon\left[(p_t' + Wp_z')/a_0^2 + \rho_0(u_r' + v_\varphi'/r + w_z' + u'/r)\right] + \varepsilon^2 d_4 = 0, \\
\rho_0 = [((\varkappa - 1)/2)\,M^2(1 - W^2) + 1]^{-1}, \quad a_0^2 = (\rho_0 M^2)^{-1},
\end{gathered} \tag{1.1}$$

where the second-order terms are

$$\begin{gathered}
d_1 = u'u_r' + u_\varphi' v'/r + w'u_z' - v'^2/r - p_r'\rho'/\rho_0^2, \\
d_2 = u'v_r' + v_\varphi' v'/r + w'v_z' + u'v'/r - p_\varphi'\rho'/r\rho_0^2, \\
d_3 = u'w_r' + w_\varphi' v'/r + w'w_z' - p_z'\rho'/\rho_0^2, \\
d_4 = \left[(\rho'/\rho_0 - p'/p)(p_t' + Wp_z') + (u'p_r' + p_\varphi' v'/r + w'p_z')\right]/a_0^2 + \rho'(u_r' + v_\varphi'/r + w_z' + u'/r).
\end{gathered}$$

The boundary conditions for the perturbations are the conditions that the perturbations must be finite in the external region and in the core flow [7].

We consider a resonant system of three waves (an axisymmetric wave and two synchronized helical waves):

$$\sum_{j=1}^{3}(u', v', w', p')_j(t, r, \varphi, z, \tau, \zeta) = \sum_{j=1}^{3} B_j(\tau, \zeta)\{u, v, w, p\}_j(r)\,e^{i\theta_j} + \text{c.c} + \varepsilon\sum_{l=1}^{3}\{\bar{u}, \bar{v}, \bar{w}, \bar{p}\}_l\,e^{i\Omega_l}. \tag{1.2}$$

Here u, v, w, p are the amplitude functions of the corresponding fluctuation quantities; $B_j(\tau, \zeta)$ are the complex wave amplitudes depending on the slow variables $\tau = \varepsilon t$ and $\zeta = \varepsilon z$, and c.c is the complex conjugate.

The phases of the waves $\theta_j = \alpha_j z + n_j\varphi - \omega_j t$ in the resonant triad are related by the phase synchronism condition $\theta_1 = \theta_2 + \theta_3$, which leads to the conditions

$$\begin{gathered}
\omega_1 = 2\omega_{2,3},\ \alpha_1 = 2\alpha_{2,3},\ n_2 = -n_3,\ n_1 = 0, \\
\Omega_1 = \theta_3 + \theta_2,\ \Omega_2 = \theta_1 - \theta_3,\ \Omega_3 = \theta_1 - \theta_2,
\end{gathered} \tag{1.3}$$

where $\alpha = \alpha^r + i\alpha^i$, α^r is the wave number and α^i is the damping decrement ($\alpha^i > 0$) or growth increment ($\alpha^i < 0$) of the perturbation.

Substituting (1.2) into (1.1) and transforming to a single equation for p_j, we obtain a recursive system of three equations relating the amplitude functions of the linear problem and the amplitudes of the waves:

$$\varepsilon\left\{\sum_{j=1}^{3}e^{i\theta_j}[B_jL(p_j)+(\partial L(p_j)/\partial\alpha_j)\partial B_j/\partial z+(\partial L(p_j)/\partial\omega_j)\partial B_j/\partial t]+\varepsilon\sum_{l=1}^{3}L(\bar{p}_l)e^{i\Omega_l}+\varepsilon\sum_{l=1}^{3}D_le^{i\Omega_l}\right\}=0. \tag{1.4}$$

Here ℓ is related to j through (1.3). In the first order in ε the system (1.4) defines the linear problem and leads to a modified Bessel's equation of order n

$$L(p_j)\equiv p_j''+(1/r-\rho_0'/\rho_0-2F_j'/F_j)p_j'+(F_j^2/a_0^2-\alpha_j^2-n_j^2/r^2)p_j=0 \tag{1.5}$$

($F_j = \alpha_j W - \omega_j$, $F_j' = \alpha_j W'$). The boundary conditions have been given in detail in [7]; a prime denotes a derivative with respect to r. The partial derivatives $\partial L/\partial\alpha$, $\partial L/\partial\omega$ needed below can be calculated from the above form of L:

$$\partial L(p_j)/\partial\alpha_j=2\left[p_j'W'\omega_j/F_j^2+p_j(F_jW/a_0^2-\alpha_j)\right],$$
$$\partial L(p_j)/\partial\omega_j=-2\left(p_j'\alpha_jW'/F_j^2+p_jF_j/a_0^2\right)$$

(D_ℓ is the nonlinear coupling coefficient of linear waves)

$$D=\left(d_1'+(1/r-2F'/F)d_1+in/rd_2+i\alpha d_3-iF/\rho_0d_4)\rho_0\right).$$

In the second order in ε we obtain equations for the second-order perturbations

$$\sum_{l=1}^{3}L(\bar{p}_le^{i\Omega_l})=-\sum_{j=1}^{3}e^{i\theta_j}((\partial B_j/\partial\zeta)\partial L(p_j)/\partial\alpha_j+(\partial B_j/\partial\tau)\partial L(p_j)/\partial\omega_j)-\sum_{l=1}^{3}e^{i\Omega_l}D_l.$$

Solutions for $\bar{p}_\ell$ exist when the right-hand sides of these equations are orthogonal to the solutions of the corresponding conjugate linear problems:

$$\sum_{j=1}^{3}\left((\partial B_j/\partial\zeta)\int_0^\infty r_j^+(\partial L(p_j)/\partial\alpha_j)\,dr+(\partial B_j/\partial\tau)\int_0^\infty p_j^+(\partial L(p_j)/\partial\omega_j)\,dr\right)+\sum_{j=l}\int_0^\infty p_j^+D_ldr=0,$$

where p_j^+ is the solution of the linear problem conjugate to (1.5)

$$p_j^{+''}-(1/r-\rho_0'/\rho_0-2F_j'/F_j)p_j^{+'}-\left[(1/r-\rho_0'/\rho_0-2F_j'/F_j)'-(F_j^2/a_0^2-\alpha_j^2-n_j^2/r^2)\right]p_j^+=0.$$

This equation is related to Bessel's equation and can be solved without difficulty.

The system of original amplitude equations for $A_j = B_je^{\gamma_jt}$ has the form [3]

$$dA_j/dt=\gamma_jA_j+\varepsilon\partial A_j/\partial\tau \tag{1.6}$$

(γ_j is the linear time constant of the oscillation). The basic problem for a jet is to find the growth of the amplitude in space. One transforms from growth in time to true growth with the help of the group velocity, which can be obtained with sufficient accuracy from the expression [4]

$$V=\int p^+(\partial L(p)/\partial\alpha)\,dr\Big/\int p^+(\partial L(p)/\partial\omega)\,dr$$

in which the limits are taken under the integral sign and $\gamma = -V\alpha^i$. The system of equations (1.6) describing the evolution of the amplitudes of the original triplet in space can be written as

$$\partial A_j/\partial z=-\alpha_j^iA_j-h\int_0^\infty p_j^+D_ldr\Big/\int_0^\infty p_j^+(\partial L(p_j)/\partial\alpha_j)dr. \tag{1.7}$$

Here for convenience in the physical interpretation of the results we introduce the trigonometric form for the complex amplitudes

$$A_j=a_je^{i\psi_j}\quad(a_j=|A_j|,\quad\psi_j=\mathrm{arctg}(A_j^i/A_j^r))$$

and we solve (1.7) for the absolute value of the amplitude a_j and the total phase $\Phi = \psi_1 - \psi_2 - \psi_3$ characterizing the relative orientation of the wave amplitudes. The constant h is the resonant coupling coefficient and takes into account the degree of phase mismatch of the resonant system. The initial value of a_j was determined by the intensity of the interacting waves $I_j = [(\langle u'^2\rangle + \langle v'^2\rangle + \langle w'^2\rangle)/3]^{1/2} \equiv B_jT_j\exp(-\alpha_j^iz)$ (T_j is the maximum mean-square fluctuation along the transverse coordinate). The longitudinal dependence of the linear wave amplitude is then $a_{j\,lin}(z) = B_j(z_0)\exp(-\alpha_j^iz)$.

2. Results and Discussion. The calculations discussed below correspond to M = 1.5 and the pair of resonance frequencies $Sh_1 = 0.25$ (axisymmetric wave) and $Sh_2 = 0.125$ (helical

waves), which correspond to [1], where for M = 1.4 the domain frequency corresponded to Sh = 0.2475. Here the acoustic Struckel number (Helmholtz) number is $\mathrm{Sh} = 2\pi\omega\bar{r}_0/a_0$ $(a_0 = (1/M^2 + (\varkappa - 1)/2)^{1/2}$ is the speed of sound outside of the jet]. The results for the wave number α^r and increment α^i are given in Fig. 1. It is evident that linear processes in the region are quite significant (large α^i) and the longitudinal variation of the axisymmetric wave (curve 1) is stronger than that of the helical waves (curve 2).

The exact conditions for resonance are satisfied in a quite narrow region in space, which should be taken into account by a reasonable choice of the resonant coupling coefficient h. Its optimal form can be obtained from the original equations, but because in the present paper we are interested in a qualitative examination of the possibilities of the resonant interaction model, we consider simple expressions for h, which are known to widen the region of resonant coupling. We consider three forms for h: $h_1 = 1 - |\Delta\alpha^r|$; $h_2 = \cos(\Delta\alpha^r z)$ (when $h_2 < 0$ we put $h_2 = 0$); the third expression, which is obviously closest to optimum, specifies the maximum value of $\Delta\alpha^r$ for which nonlinear coupling is possible:

$$\text{when } \Delta\alpha^r \leqslant 0{,}2 \quad h_3 = \cos\left(\frac{\pi}{2}\frac{\Delta\alpha^r}{0{,}2}\right), \quad \text{when } \Delta\alpha^r > 0{,}2\,(\delta > 0{,}6) \quad h_3 = 0.$$

Qualitative differences in the form and nature of the interactions were not found. The most suitable form is obviously of type h_3 with the limiting value $\Delta\alpha^r$ chosen to correspond to the experimental data.

The longitudinal dependence of the nonlinear coefficient

$$K = \int\limits_0^6 p^+ D dr \Bigg/ \int\limits_0^6 p^+ (\partial L(p)/\partial\alpha)\, dr$$

is shown in Fig. 2 (the notation is the same as in Fig. 1). Note that the real and imaginary parts of K_j are quite complicated functions of the same order and typically change sign twice in the interval of z considered here.

We consider three typical cases ($I_2 = I_3 > I_1$, $I_2 = I_3 < I_1$, $I_3 < I_2 < I_1$).

1. Intensity of the Helical Waves Exceeds the Intensity of the Axisymmetric Wave. The competing mechanism in this type of energy transfer (pumping of energy into the double-frequency wave) is pair interaction where (in the simplest case) an overtone at frequency $\omega_1 = 2\omega_2$ is induced by the self-action of the waves at frequency ω_2. The mode of the induced wave is preserved and the interaction itself is second order in the resonance. Obviously the primary problem is to examine this model and to compare its effectiveness with the resonant model, which could also help determine the type of oscillations in the peaks.

An important quantity in this case is the minimum intensity at which the nonlinear interaction can be distinguished from the linear. The threshold value is $I_2 \sim 2\%$, which corresponds to the initial amplitude $a_2 \sim 10^{-3}$.

The magnitude and directivity of the nonlinear wave interaction depends in an essential way on the relative orientation of the amplitude wave vectors A_j, which is determined by the initial value of the total phase Φ_0. The transfer from the helical waves to the axisymmetric wave is maximum for Φ_0 in the third quadrant $(220° \leqslant \Phi_0 \leqslant 230°)$ for any intensities of the interacting oscillations. This is illustrated in Fig. 3. The initial intensity of the helical waves was chosen to be large enough ($I_2 = 15\%$) to show clearly all of the features of the process, although similar features are present in lower-intensity oscillations exceeding the threshold. The solid curves show the longitudinal variation of the axisymmetric wave amplitude a_1 for different initial values, where curves 1-7 refer to the intensity levels $I_1 = I_2/k$ with k = 200, 100, 50, 20, 10, 5, 2. The dashed curves show the linear growth of a_1 for waves of the indicated intensity level. We see that the effect of the nonlinearity increases with increasing difference between the initial intensities of the interacting waves. As the two intensities approach one another the contribution of nonlinearity decreases (compare curves 1, 4, and 7 corresponding to $I_1 = I_2/200$, $I_2/20$, $I_2/2$), i.e., background oscillations can be effectively amplified because of the effect of nonlinearity.

Overall the interaction is regular and does not display the catastrophic amplitude growth typical of a subsonic boundary layer. It corresponds to a redistribution of energy between the resonant oscillations. Figure 4 shows the behavior of the helical wave amplitudes

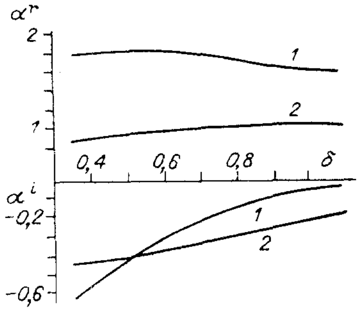

Fig. 1

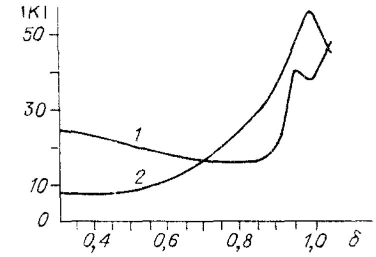

Fig. 2

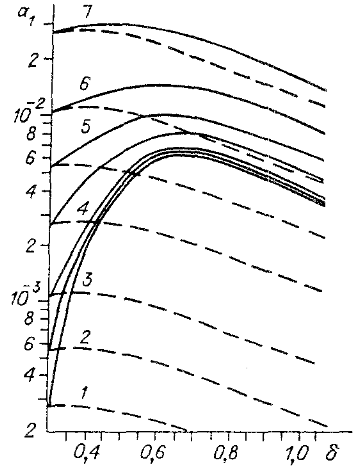

Fig. 3

$a_2 = a_3$ corresponding to intensity levels 6 and 7 and the same Φ_0. It is evident that the behavior of a_2 changes only slightly from linear.

The phases of maximum and minimum transfer differ by 180° and the latter is in the first quadrant $(40° \leqslant \Phi_0 \leqslant 50°)$. Figure 5 shows a_1 at $\Phi_0 = 55°$ and the initial I_1 values of Fig. 3. We see that for background (or sufficiently small) a_1 (curves 1-3) the amplification of the axisymmetric wave exceeds the linear level, except for a small initial region in z. But as the initial values of a_2 and a_1 approach one another the region of space expands where the amplification is below the linear level and when $a_1 \sim 4 \cdot 10^{-3}$ the directivity of the entire process changes. The energy from an axisymmetric wave of lower intensity is converted into higher intensity helical waves and their intensities increase and begin to exceed the linear values (Fig. 4).

Values of Φ_0 differing from the minmax phases by 90° (second and fourth quadrants) are cutoff phases. Only small background oscillations are amplified and the behavior of a_1 and a_2 is determined mainly by the linear laws of growth.

Hence, we conclude that if the wave amplitudes differ strongly energy is transferred to the higher frequency for arbitrary phase orientation of the amplitude vectors but as the intensities approach one another specific "phase rules" determine the directivity of the transfer process.

2. Intensity of the Axisymmetric Wave Exceeds the Intensity of the Helical Waves. This mechanism corresponds to the C mechanism of transfer (excitation of subharmonics) in a subsonic boundary layer. All of the typical features of this type of interaction are illustrated in Fig. 6 (I_2 = 1.5% and 7.5% with I_1 = 15%). Here again the process deviates from linear when $I_1 \sim 2\%$. The process is again regular and depends on the relative phase orientation of the amplitude vectors of the waves. The most favorable phase for excitation of helical waves is $\Phi_0 \sim 5°$ (curve 1) and when $\Phi_0 \sim 185°$ the level a_2 decreases for arbitrary relations between the amplitudes (in contrast to the case discussed above) (curve 2). The cutoff phases of the nonlinear process (curve 3) are $\Phi_0 \sim 125°$ and 235°. Transfer occurs by parametric amplification and is determined purely by the initial value of I_1. For example, when $I_1 = 5\%$ the increase or decrease of a_2 (for all I_2 between $I_1/200$ and $I_1/2$) is by a factor of 1.4; for $I_1 = 10\%$ the factor is 2, for $I_1 = 15\%$ it is 2.8. We note that in general the transfer of energy in the direction of the lower frequency is weaker and depends more on the linear process.

3. Intensities of All Waves are Different. This case includes a very large number of different combinations. The situation $I_3 < I_2 < I_1$ is the most interesting because in this case one can model the appearance of synchronized beating oscillations in the spectrum. This case is shown in Fig. 7 for wave intensities in multiples of 10: $I_1 = 15\%$, $I_2 = 1.5\%$, $I_3 = 0.15\%$. We see that none of the restrictions on the phase of the interaction hold, although $\Phi_0 \sim 0$ remains the phase of maximum transfer to lower frequencies. Secondly it is evident that the "capture" phenomenon, accompanied by a rapid equalization of the amplitudes of the helical waves, does not occur. Figure 7 shows that the maximum approach of the amplitudes occurs at $\Phi_0 = 5°$, where the difference between them is still large (curve 1). At $\Phi_0 = 185°$ (for equal a_2 and a_3 this phase gives transfer back to the axisymmetric wave) there is

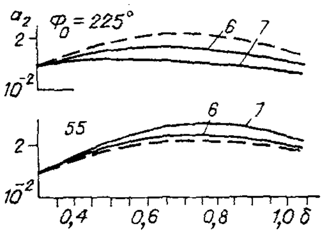

Fig. 4

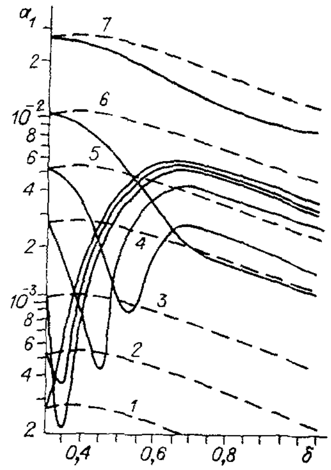

Fig. 5

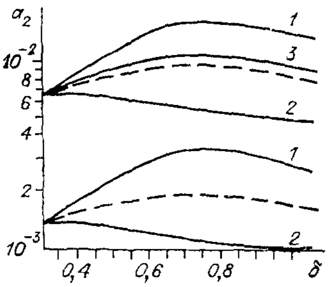

Fig. 6

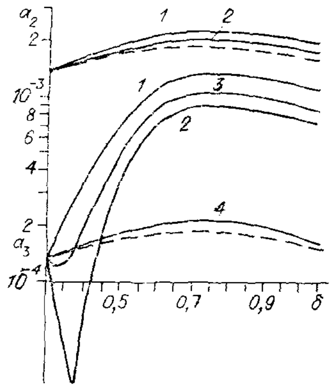

Fig. 7

also an increase in a_3 (curve 2) accompanied by a decrease in a_2. The calculations show that approach of a_2 and a_3 is more probable if their initial values are not too different, but complete equalization of a_2 and a_3 is not observed. Figure 7 also shows a_3 at the cut-off phases Φ_0 = 125° and 255° (curves 3 and 4). Here a_2 is close to the level corresponding to Φ_0 = 185°. In all of these cases the amplitude a_1 is close to the linear value.

Obviously the significant decrease in the level of excitation of a_2 in comparison with case 2 is due to the fact that nonlinear coupling causes some of the energy to go into amplifying the less intense wave a_3. The absence of the "capture" phenomenon serves as indirect support that the interaction process is indeed slightly nonlinear.

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